

Kinematic self-similar locally rotationally symmetric models

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Abstract

A brief summary of results on kinematic self-similarities in general relativity is given. Attention is focussed on locally rotationally symmetric models admitting kinematic self-similar vectors. Coordinate expressions for the metric and the kinematic self-similar vector are provided. Einstein's field equations for perfect fluid models are investigated and *all* the homothetic perfect fluid solutions admitting a maximal four-parameter group of isometries are given.

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1 Introduction

This article is focused on kinematic self-similar models exhibiting a maximal four-parameter group of isometries, G_4 (in addition to the self-similar symmetry). Only perfect fluid models are considered.

Perfect fluid solutions admitting a maximal simply-transitive group G_4 are all known. They correspond to the homogeneous Ozsváth solutions [1]. In the multiply-transitive case, they are locally rotationally symmetric (LRS), the maximal group G_4 acts on three-dimensional non-null orbits S_3 or T_3 , and the solutions are algebraically special. Such models are the concern of this paper. Their metrics can be written in the forms [2, 3]

$$(i) \quad ds^2 = \epsilon(dt^2 - A^2(t)dx^2) + B^2(t)(dy^2 + \Sigma^2(y, k)dz^2), \quad (1)$$

$$(ii) \quad ds^2 = \epsilon(dt^2 - A^2(t)\sigma^2) + B^2(t)(dy^2 + \Sigma^2(y, k)dz^2), \quad (2)$$

$$(iii) \quad ds^2 = \epsilon(dt^2 - A^2(t)dx^2) + B^2(t)e^{2x}(dy^2 + dz^2), \quad (3)$$

with $\epsilon = \pm 1$, $k = 0, \pm 1$ and $\sigma = dx + \Gamma(y, k)dz$, where

$$\Gamma(y, k) = \begin{cases} \cos y & k = +1 \\ y^2 & k = 0 \\ \cosh y & k = -1 \end{cases}, \quad \Sigma(y, k) = \begin{cases} \sin y & k = +1 \\ y & k = 0 \\ \sinh y & k = -1 \end{cases}. \quad (4)$$

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The known exact solutions of the metric (i) with $\epsilon = -1$ have been collected by Vajk and Eltgroth [4], and the field equations for the metric (iii) have been qualitatively studied by Collins [5] and Shikin [6]. General LRS space-times have been investigated many times in the literature [7]. The symmetry groups of the LRS models have been discussed by Stewart and Ellis [3], van Elst and Ellis [8], van Elst and Uggla [9]. More recently, LRS perfect fluids have been studied by Marklund [10], and Nilsson and Uggla [11].

In this paper, we study LRS perfect fluids (with a maximum four-parameter group of isometries) admitting kinematic self-similar vectors.

The concept of kinematic self-similarity was first introduced by Carter and Henriksen [12, 13] as a generalization to the homothety [14] and as a more natural counterpart of the concept of self-similarity present in Newtonian mechanics.

A vector field X is called a kinematic self-similar vector field (KSS) if it satisfies the conditions [12, 13]

$$\mathcal{L}_X u_a = \alpha u_a , \quad \mathcal{L}_X h_{ab} = 2\delta h_{ab} , \quad (5)$$

where α and δ are constants, \mathcal{L} stands for the Lie derivative operator, u^a is the four-velocity of the fluid, and $h_{ab} = g_{ab} + u_a u_b$ is the projection tensor of the metric into the three-spaces orthogonal to u^a .

The kinematic self-similar transformations are characterized in a well defining way by the scaling-independent ratio α/δ which is referred to as the similarity index. This index is finite except in the case of rigid transformations which is characterized by $\delta = 0$ and referred to as type infinite. In the case $\alpha = \delta$, it follows that X is a homothetic vector field (HVF) [14] and, evidently, if $\alpha = \delta = 0$, then X becomes a Killing vector (KV). Another case of special interest is the type zero (i.e., $\alpha = 0$) which corresponds to space dilatation without time amplification.

The study of self-similar models is very important in general relativity. Some self-similar models are tractable because their symmetry makes them less complicated. Besides their intrinsic mathematical interest, self-similar solutions play an important role in astrophysics and cosmology (see [15, 16] for a detail review). The astrophysical applications include gravitational collapse and the occurrence of naked singularities. The cosmological applications include features of gravitational clustering, cosmic voids, the formation of bubbles at a cosmological phase-transition in the early universe, explosions in an (expanding) homogeneous background, cosmological models containing black holes, and their possible role as asymptotic states for more general models.

The rest of the paper is organized as follows. The next section contains a brief summary of results on kinematic self-similar vector fields and space-times admitting them. In section 3, we analyze LRS models constrained to admit KSS and we provide coordinate expressions for the metric and the KSS. In section 4, we present the different perfect fluid solutions. The study is exhausted for the metrics (i) and (ii). For the metric (iii), we distinguish the cases where the fluid flow is comoving from where it is non-comoving. In the comoving case, the only possible perfect fluid solution is a special case of a Friedmann-Robertson-Walker (FRW) model and, in the non-comoving case, we show that perfect fluid solutions are non-homothetic. Moreover, since the case of simply-transitive group G_4 is empty of homothetic solutions [17, 18], we explicitly give *all* the homothetic solutions admitting a maximal 4-dimensional group of isometries along with the kinematic quantities characterizing the fluid (see tables 11-14). Finally, in section 5, we discuss the results obtained.

2 Kinematic self-similarity

In this section we present a brief summary of results on kinematic self-similar vector fields and space-times admitting them.

1. The set of all KSS of the space-time forms a finite-dimensional Lie algebra under the usual Lie bracket operation and will be denoted by K_n , where n is its dimension. Furthermore, it can be seen by direct computation that the Lie bracket of two arbitrary KSS is always a KV.
2. The set of all KSS with the same similarity index, $\kappa = \alpha/\delta$, forms also a finite-dimensional Lie algebra, which will be denoted by K_s^κ (where s is its dimension), and one has that $K_s^\kappa \subseteq K_n$ (i.e., K_s^κ is a subalgebra of K_n). Each non trivial K_s^κ algebra, in the sense that a proper-KSS (not a KV) exists, contains an $(s - 1)$ -dimensional subalgebra of KV. Equivalently, given two proper-KSS with the same similarity index κ , there always exists a linear combination of them which is a KV. The particular case K_s^1 (i.e., $\alpha = \delta$) corresponds to the homothetic algebra.
3. Given two KSS, X_1 and X_2 ,

$$\begin{aligned}\mathcal{L}_{X_1} h_{ab} &= 2\delta_1 h_{ab}, & \mathcal{L}_{X_1} u_a &= \alpha_1 u_a, \\ \mathcal{L}_{X_2} h_{ab} &= 2\delta_2 h_{ab}, & \mathcal{L}_{X_2} u_a &= \alpha_2 u_a,\end{aligned}$$

there always exist two vectors defined by

$$Y \equiv \alpha_2 X_1 - \alpha_1 X_2, \quad Z \equiv \delta_2 X_1 - \delta_1 X_2,$$

so that Y is a KSS of type zero and Z is a KSS of type infinite.

4. If two proper-KSS (not KV) of different similarity index exist in the space-time, then any other KSS is a linear combination of them and the KVs. From this consideration, it immediately follows that if the dimension of the isometric algebra of the space-time is r , then the maximum dimension of the kinematic self-similar algebra is $r + 2$. In this case, one can always construct two proper-KSS, such as one is of type zero and the other is of type infinite. Furthermore, the space-time always admits an $(r + 1)$ -dimensional homothetic algebra.

Proof: Let $X_i \in K$, so that $\mathcal{L}_{X_i} h_{ab} = 2\delta_i h_{ab}$, $\mathcal{L}_{X_i} u_a = \alpha_i u_a$, $i = 1, 2, 3$ and $\alpha_2\delta_1 - \alpha_1\delta_2 \neq 0$, then $V \equiv ((\alpha_2 - \delta_2)X_1 + (\delta_1 - \alpha_1)X_2)/(\alpha_2\delta_1 - \alpha_1\delta_2)$ is a HVF, and $W \equiv (\alpha_2\delta_1 - \alpha_1\delta_2)X_3 + (\alpha_3\delta_2 - \alpha_2\delta_3)X_1 + (\alpha_1\delta_3 - \alpha_3\delta_1)X_2$ is a KV.

5. For a four-dimensional manifold, the highest possible dimension of a kinematic self-similar algebra is $n = 12$, and can only occur (as well as for $n = 11$) when the connection is flat and obviously no perfect fluid solution exists.

The case $n = 10$ is impossible, as it follows from considerations of the dimension of the isometric algebra.

If $n = 9$ then, the Killing subalgebra is 7-dimensional and multiply transitive acting on the 4-dimensional manifold. The resulting 3-parameter Killing isotropy then implies that the Weyl tensor has Petrov type O , and since the space-times must admit a homothetic algebra, known results regarding fixed points of the homothety [18] implies that the Ricci tensor has Segre type $\{(2, 11)\}$. It then follows that the space-time is a conformally flat homogeneous generalized plane wave (the “null-fluid” case) and so, there cannot be perfect fluid solutions.

3 Analysis

In this section, we study the different LRS space-times admitting a four-dimensional group of isometries, and we provide coordinate expressions for the metric and the kinematic self-similar vector field.

Given the coordinate forms of the KVs for the metrics (1)-(3), assuming the existence of a KSS, X , since its commutator with a KV must be a KV, the Jacobi identities imply that, in these coordinates, the KSS can only take the forms given in table 1,

| case | Killing vectors | KSS, X |
|-------|--|--|
| (i) | $\sin z\partial_y + \sum' \cos z\partial_z$ $\cos z\partial_y - \sum' \sin z\partial_z$ ∂_x ∂_z | $X^t(t)\partial_t + (\hat{X}^x(t) + nx)\partial_x + my\partial_y$ $m, n \in \mathbb{R}$ $m = 0$ for $k \neq 0$ |
| (ii) | $\cos z\Delta\partial_x + \sin z\partial_y + \cos z\sum'\partial_z$ $\sin z\Delta\partial_x - \cos z\partial_y + \sin z\sum'\partial_z$ ∂_x ∂_z | $X^t(t)\partial_t + (\hat{X}^x(t) + 2mx)\partial_x + my\partial_y$ $m \in \mathbb{R}$ $m = 0$ if $k \neq 0$ |
| (iii) | $\partial_x - y\partial_y - z\partial_z$ $z\partial_y - y\partial_z$ ∂_y ∂_z | $X^t(t)\partial_t + X^x(t)\partial_x$ |

Table 1. KVs and possible KSS for the different LRS metrics (i), (ii) and (iii).

where a prime indicates a derivative with respect to y and

$$\Delta = \Gamma \left(\frac{\Gamma'}{\Gamma} - \frac{\Sigma'}{\Sigma} \right) . \quad (6)$$

From now on, we will devote our study only to those space-times which can be interpreted as perfect fluid solutions. The energy momentum tensor, T_{ab} , is given by

$$T_{ab} = (\mu + p)u_a u_b + pg_{ab} , \quad (7)$$

where μ is the energy density, p the pressure, and u^a the four-velocity of the fluid.

By simple inspection of the field equations, one can figure out which components of the four-velocity vanish. Then, solving the kinematic self-similar conditions (5) in the covariant unknowns u_a and h_{ab} for the contravariant X given in table 1, we obtain restrictions for X and, depending on the value of the parameters ϵ , α and δ , we obtain the different forms of the metric functions in each case.

Case (i)

The possible forms of the four-velocity u and the KSS X for the line element (1) are those given in table 2,

| (i) | u | X |
|-----------------|-----------|---|
| $\epsilon = -1$ | $-dt$ | $(\alpha t + \beta) \partial_t + nx \partial_x + my \partial_y$ |
| $\epsilon = +1$ | $-A(t)dx$ | $(\delta t + \beta) \partial_t + nx \partial_x + my \partial_y$ |

Table 2. Four-velocity and KSS for the metric (1).

where β is a constant.

For $\epsilon = -1$ and $k = 0$, the space-time (1) always admits a proper-KSS, say $\bar{X} = x \partial_x + y \partial_y$, of type zero independently of the form of the metric functions. It corresponds to the case $\alpha = \beta = 0$. Apart from this case, the following possibilities arise

| $\epsilon = -1$ | case | $A(t)$ | $B(t)$ |
|-----------------|----------------------------|--|--|
| (i.a) | $\alpha \neq 0, \beta = 0$ | $t^{(\delta-n)/\alpha}$ | $b t^{(\delta-m)/\alpha}$ |
| (i.b) | $\alpha = 0, \beta \neq 0$ | $\exp\left(\frac{\delta-n}{\beta}t\right)$ | $b \exp\left(\frac{\delta-m}{\beta}t\right)$ |

Table 3. Metric functions for the line element (1) with $\epsilon = -1$.

where b is a constant. In the case (i.a) since $\alpha \neq 0$, by a translation of t , we can set β to zero and, by rescaling X with a factor $1/\alpha$, we can set the parameter α to unity. In the case (i.b), we can also set β to 1. Note that, for $k = 0$, the case (i.a) admits two proper-KSS, X and \bar{X} , of different similarity index and therefore, the space-time is homothetic and, in the case (i.b), since X and \bar{X} are two independent KSS of type zero, the group G_4 is not maximal.

For $\epsilon = +1$, the vector $\hat{X} = x \partial_x$ is a proper-KSS of type infinite. It satisfies $\mathcal{L}_{\hat{X}} u_a = u_a, \mathcal{L}_{\hat{X}} h_{ab} = 0$ for all $A(t)$ and $B(t)$. The remaining possibilities are given in table 4.

| $\epsilon = +1$ | case | $A(t)$ | $B(t)$ |
|-----------------|----------------------------|--|--|
| (i.c) | $\delta \neq 0, \beta = 0$ | $t^{(\alpha-n)/\delta}$ | $b t^{(\delta-m)/\delta}$ |
| (i.d) | $\delta = 0, \beta \neq 0$ | $\exp\left(\frac{\alpha-n}{\beta}t\right)$ | $b \exp\left(-\frac{m}{\beta}t\right)$ |

Table 4. Metric functions for the line element (1) with $\epsilon = +1$.

Again, by normalizing X we can set δ or β to 1. Note that the case (i.c) admits also a proper-HVF, and in the case (i.d), there exists a further KV and G_4 is not maximal.

Case (ii)

The four-velocity and the KSS for the line element (2) must have the forms given in table 5.

| (ii) | u | X |
|-----------------|-------------------------|--|
| $\epsilon = -1$ | $-dt$ | $(\alpha t + \beta) \partial_t + 2mx \partial_x + my \partial_y$ |
| $\epsilon = +1$ | $-A(t)(dx + \Gamma dz)$ | $(\delta t + \beta) \partial_t + 2mx \partial_x + my \partial_y$ |

Table 5. Four-velocity and KSS for the metric (2).

The only possible cases are those listed in tables 6 and 7,

| $\epsilon = -1$ | case | $A(t)$ | $B(t)$ |
|-----------------|----------------------------|---|--|
| (ii.a) | $\alpha \neq 0, \beta = 0$ | $a t^{(\delta-2m)/\alpha}$ | $b t^{(\delta-m)/\alpha}$ |
| (ii.b) | $\alpha = 0, \beta \neq 0$ | $a \exp\left(\frac{\delta-2m}{\beta}t\right)$ | $b \exp\left(\frac{\delta-m}{\beta}t\right)$ |

Table 6. Metric functions for the line element (2) with $\epsilon = -1$.

| $\epsilon = +1$ | case | $A(t)$ | $B(t)$ |
|-----------------|----------------------------|---|--|
| (ii.c) | $\delta \neq 0, \beta = 0$ | $a t^{(\alpha-2m)/\delta}$ | $b t^{(\delta-m)/\delta}$ |
| (ii.d) | $\delta = 0, \beta \neq 0$ | $a \exp\left(\frac{\alpha-2m}{\beta}t\right)$ | $b \exp\left(-\frac{m}{\beta}t\right)$ |

Table 7. Metric functions for the line element (2) with $\epsilon = +1$.

where a and b are constants. In each case, we can set α, β or δ to unity respectively.

Case (iii)

By simple inspection of the field equations for the metric (3), we obtain

$$u = u_t(t)dt + u_x(t)dx , \quad (8)$$

subject to the condition

$$u_t(t)^2 = -\epsilon + \frac{u_x(t)^2}{A(t)^2} . \quad (9)$$

The kinematic self-similar equations (5) specified to the KSS given in table 1 yield

$$u_{x,t}X^t = \alpha u_x , \quad (10)$$

$$u_{t,t}X^t = \epsilon(\delta - \alpha)u_t \frac{u_x^2}{A^2} , \quad (11)$$

$$X^t,_t = \delta + \epsilon(\delta - \alpha)u_t^2 , \quad (12)$$

$$X^x,_t = -2\epsilon(\delta - \alpha)\frac{u_t u_x}{A^2} , \quad (13)$$

$$(A^2),_t X^t = 2\delta A^2 - 2\epsilon(\delta - \alpha)u_x^2 , \quad (14)$$

$$(B^2),_t X^t + 2B^2 X^x = 2\delta B^2 . \quad (15)$$

We study now two different cases:

Comoving fluid flow

In this case the component G_{tx} of the Einstein tensor must vanish. This yields

$$A(t) = aB(t) , \quad a \in \mathbb{R} , \quad (16)$$

and the following possibilities arise

| (iii) | u | X |
|-----------------|-----------|--|
| $\epsilon = -1$ | $-dt$ | $(\alpha t + \beta)\partial_t + n\partial_x$ |
| $\epsilon = +1$ | $-A(t)dx$ | $(\delta t + \beta)\partial_t + n\partial_x$ |

Table 8. Four-velocity and KSS for the metric (3) with a comoving fluid flow.

where β and n are constants. For $\epsilon = -1$ the functional forms of the metric functions are given in table 9. We note that, in all cases, the constant n must vanish, and the solutions are special cases of FRW models.

| $\epsilon = -1$ | case | $B(t)$ | constraints |
|-----------------|----------------------------|--|-------------|
| (iii.a) | $\alpha \neq 0, \beta = 0$ | $t^{\delta/\alpha}$ | $n = 0$ |
| (iii.b) | $\alpha = 0, \beta \neq 0$ | $\exp\left(\frac{\delta}{\beta}t\right)$ | $n = 0$ |

Table 9. Metric functions for the line element (3) with $\epsilon = -1$ in the comoving case.

For $\epsilon = +1$ the metric functions are those given next in table 10.

| $\epsilon = +1$ | case | $B(t)$ | constraints |
|-----------------|----------------------------|--------------------------------------|-----------------------|
| (iii.c) | $\delta \neq 0, \beta = 0$ | $t^{(\delta-n)/\delta}$ | $\alpha = \delta - n$ |
| (iii.d) | $\delta = 0, \beta \neq 0$ | $\exp\left(-\frac{n}{\beta}t\right)$ | $\alpha = -n$ |

Table 10. Metric functions for the line element (3) with $\epsilon = +1$ in the comoving case.

Non-comoving fluid flow

In this case $u_t \neq 0, u_x \neq 0$ and $A(t) \neq aB(t)$. We distinguish two cases whereas $u_{t,t}$ vanish or not. The case $u_{t,t} = 0$ can be fully integrated. Demanding X not to be a KV, it yields

$$u = \beta dt + ctdx, \quad X = t\partial_t + n\partial_x, \quad \beta, c, n \in \mathbb{R} \quad (17)$$

$$A(t) = \frac{ct}{\sqrt{\beta^2 + \epsilon}}, \quad B(t) = t^{1-n}, \quad (18)$$

which is a homothetic space-time with $\alpha = \delta = 1$.

Let us consider now the case $u_{t,t} \neq 0$. From equations (9), (11) and (12), we obtain

$$-\alpha + 2\epsilon(\delta - \alpha)u_t^2 - \epsilon(\delta - \alpha)u_t(u_t^2 + \epsilon)\frac{u_{t,t}}{(u_{t,t})^2} = 0. \quad (19)$$

Hence, it readily follows that proper homothetic solutions are not possible. For $\alpha \neq \delta$, equation (19) can be integrated once and we get

$$(u_t^2)_{,t} = c(u_t^2)^{\frac{\delta/2-\alpha}{\delta-\alpha}}(u_t^2 + \epsilon)^{\frac{\delta-\alpha/2}{\delta-\alpha}}, \quad c \in \mathbb{R}. \quad (20)$$

Explicit solutions can be obtained for some particular values of the parameters α and δ but, in general, the solution of equation (20) will be given in an implicit form. The remaining quantities can be obtained in terms of u_t

$$X^t = \frac{2\epsilon}{c}(\delta - \alpha)(u_t^2)^{\frac{\delta/2}{\delta-\alpha}}(u_t^2 + \epsilon)^{\frac{-\alpha/2}{\delta-\alpha}}, \quad (21)$$

$$u_x = e \left(\frac{u_t^2}{u_t^2 + \epsilon} \right)^{\frac{\alpha}{2(\delta-\alpha)}}, \quad e \in \mathbb{R}, \quad (22)$$

$$A^2 = \frac{u_x^2}{u_t^2 + \epsilon}. \quad (23)$$

Then, integrating (13) we get X^x , and from (15) we obtain $B(t)$.

4 Perfect fluid solutions

In this section we present the different perfect fluid solutions admitting a KSS with a maximal group G_4 of isometries with an explicit mention of the physical quantities characterizing the fluid; namely: density μ , γ (appearing in the equation of state $p = (\gamma - 1)\mu$), acceleration \dot{u}_a , vorticity ω_{ab} , shear σ , volume expansion θ , and deceleration parameter q . In what follows, we have eliminated some of the parameters, in order to simplify the expressions.

Case (i), $\epsilon = -1$.

The matter variables are

$$\mu = 2 \frac{\dot{A}\dot{B}}{AB} + \frac{k}{B^2} + \left(\frac{\dot{B}}{B} \right)^2, \quad p = -2 \frac{\ddot{B}}{B} - \frac{k}{B^2} - \left(\frac{\dot{B}}{B} \right)^2, \quad (24)$$

where a dot indicates a derivative with respect to t and the non-trivial field equation is

$$\frac{\ddot{B}}{B} + \frac{k}{B^2} + \left(\frac{\dot{B}}{B} \right)^2 - \frac{\dot{A}\dot{B}}{AB} - \frac{\ddot{A}}{A} = 0. \quad (25)$$

The perfect fluid solutions for the case (i.a) are given in table 11.

| (i.a) | μ | γ | δ | m | b^2 | σ^2 | θ | q |
|----------|----------------------------|-----------------|----------|-----------------|----------------------|--------------------------|-----------------|-----------------|
| $k = 1$ | $\frac{n^2 - 4n + 3}{t^2}$ | $\frac{2}{3-n}$ | 1 | 0 | $\frac{1}{n^2 - 2n}$ | $\frac{n^2}{3t^2}$ | $\frac{3-n}{t}$ | $\frac{n}{3-n}$ |
| $k = 0$ | $\frac{n(4-3n)}{2t^2}$ | 2 | 1 | $\frac{2-n}{2}$ | 1 | $\frac{(2-3n)^2}{12t^2}$ | $\frac{1}{t}$ | 2 |
| $k = -1$ | $\frac{n^2 - 4n + 3}{t^2}$ | $\frac{2}{3-n}$ | 1 | 0 | $\frac{1}{2n - n^2}$ | $\frac{n^2}{3t^2}$ | $\frac{3-n}{t}$ | $\frac{n}{3-n}$ |

Table 11 Perfect fluid solutions for the case (i.a).

For all the above cases we have $\dot{u}_a = \omega_{ab} = 0$ (geodesic and irrotational flow) and X is a proper-HVF. All of them are special cases of Szekeres-Szafron universes [19]. There exists another possible solution for $k = 0$ and $n = m$, but this is a special type of FRW model and, in keeping with the assumption of maximality of G_4 , we have not listed it here. Also, notice that the case $k = 0$, for the particular value $m = \frac{2}{3}$, corresponds to an FRW model as well.

For the case (i.b) there is no solution for $k = -1$; for $k = 1$, X becomes a KV; and for $k = 0$, the group G_4 is not maximal.

The only possible proper kinematic self-similar solution (not homothetic), for the metric (1) with $\epsilon = -1$, corresponds to $k = 0$ without any further assumption a priori for the metric functions. Thus, integrating equation (25) for $k = 0$, we obtain

$$A(t) = c_1 B(t) + c_2 B(t) \int \frac{dt}{B(t)^3}, \quad (26)$$

where c_1 and c_2 are constants and $B(t)$ is an arbitrary function. For an equation of state $p = (\gamma - 1)\mu$, the solution for $\gamma \neq 2$ becomes [3]

$$A = (B^{3(2-\gamma)/2} + c)^{1/(2-\gamma)} B^{-1/2},$$

$$t = \int (B^{3(2-\gamma)/2} + c)^{(\gamma-1)/(2-\gamma)} B^{1/2} dB , \quad (27)$$

$$\mu = \frac{3}{(AB^2)^\gamma} ,$$

where c is a constant, and for $\gamma = 2$ it corresponds to the homothetic solution listed in **Table 11**.

Case (i), $\epsilon = +1$.

The field equations reduce to

$$\mu = -2\frac{\ddot{B}}{B} + \frac{k}{B^2} - \left(\frac{\dot{B}}{B}\right)^2 , \quad p = 2\frac{\dot{A}\dot{B}}{AB} - \frac{k}{B^2} + \left(\frac{\dot{B}}{B}\right)^2 , \quad (28)$$

and

$$\frac{\ddot{B}}{B} + \frac{k}{B^2} - \left(\frac{\dot{B}}{B}\right)^2 - \frac{\dot{A}\dot{B}}{AB} + \frac{\ddot{A}}{A} = 0 , \quad (29)$$

Since $\hat{X} = x\partial_x$ is always a proper-KSS, any pair of functions $A(t)$ and $B(t)$ satisfying (29) will represent a kinematic self-similar solution if they satisfy the energy conditions. The only solutions corresponding to homothetic space-times are listed in table 12.

| (i.c) | μ | γ | α | m | b^2 |
|---------|--------------------------------------|-----------------|----------|---------------------|-------------------|
| $k = 1$ | $\frac{1-n^2}{t^2}$ | $\frac{2}{1+n}$ | 1 | 0 | $\frac{1}{2-n^2}$ |
| $k = 0$ | $\frac{n(1-n^2)(3n-4)}{(2-n)^2 t^2}$ | $\frac{2}{1+n}$ | 1 | $\frac{2-n^2}{2-n}$ | 1 |

Table 12 Perfect fluid solutions for the case (i.c).

Note that t here is a spacelike coordinate and all these solutions are stationary. They have $\sigma = \theta = \omega_{ab} = 0$, whereas $\dot{u}_a = \frac{1-n}{t}\delta_a^t$. The case $k = 1$ is further discussed in [20, 21, 22] corresponding to a static spherically symmetric solution. No physically realistic solutions exist for $k = -1$ since the energy density is then negative necessarily.

Case (ii), $\epsilon = -1$.

The only possible kinematic self-similar solution with a maximal group G_4 is given in table 13.

| (ii.a) | μ | γ | δ | $a^2 b^{-4}$ |
|---------|----------------------------|----------------------------|----------|---------------------|
| $k = 0$ | $\frac{6-17m+12m^2}{2t^2}$ | $\frac{4-6m}{6-17m+12m^2}$ | 1 | $\frac{m}{2} - m^2$ |

Table 13 Perfect fluid solution for the case (ii.a).

It corresponds to a homothetic solution. The fluid is geodesic, irrotational and

$$\sigma^2 = \frac{m^2}{3t^2}, \quad \theta = \frac{3-4m}{t}, \quad q = \frac{4m}{3-4m}. \quad (30)$$

The case $k = 1$ corresponds again to a special type of FRW model with equation of state $\mu + 3p = 0$, whereas no physically significant solutions exist for $k = -1$, since the energy density is always negative.

Case (ii.b) is empty of solutions with a maximal group G_4 . The case $k = 1$ corresponds to an FRW model and there are no perfect fluid solutions in the other cases.

Case (ii), $\epsilon = +1$.

In table 14, we give all perfect fluid solutions.

| (ii.c) | μ | γ | α | a^2 |
|----------|--------------------------------|-------------------------|----------|------------------|
| $k = 1$ | $\frac{2-\frac{1}{2b^2}}{t^2}$ | 2 | 1 | $2b^2(2b^2 - 1)$ |
| $k = 0$ | $\frac{2+m-6m^2}{t^2}$ | $\frac{4-6m}{2+m-6m^2}$ | 1 | $(1-m-m^2)b^4$ |
| $k = -1$ | $\frac{2+\frac{1}{2b^2}}{t^2}$ | 2 | 1 | $2b^2(2b^2 + 1)$ |

Table 14 Perfect fluid solutions for the case (ii.c).

Note that all of them are stationary and homothetic with

$$\dot{u}_a = \frac{1-2m}{t} \delta_a^t, \quad \sigma = \theta = 0. \quad (31)$$

The only non-vanishing component of the vorticity tensor is

$$\omega_{yz} = \frac{a}{2} t^{1-2m} \Gamma'. \quad (32)$$

For the case (ii.d) there are no solutions for $k = 1$ and $k = 0$, and for $k = -1$, X is a KV therefore, the G_4 is not maximal.

Case (iii)

Comoving fluid flow.

The only possible solutions of Einstein's field equations for a perfect fluid correspond in this case to $\epsilon = -1$, case (iii.a), the solution being again a special type of FRW model with an equation of state $\mu + 3p = 0$. For $\epsilon = +1$, (iii.c) corresponds to a homothetic vacuum space-time, and there is no solution for the case (iii.d).

Non-comoving fluid flow.

The Einstein's field equations for the case $u_{t,t} = 0$ (i.e., solution (18)) imply $\beta^2 + \epsilon = c^2(1-n)^2$ and hence, $\mu + p = 0$. Therefore, no perfect fluid exists with $\mu + p > 0$.

The case in which $u_{t,t} \neq 0$, homothetic solutions are not possible. Furthermore, given the solution of equation (20) one has to check the existence or not of perfect fluid solutions for the different value of the parameters α and δ .

5 Discussion

To summarize, we have reviewed the concept of kinematic self-similar vector fields, presented some results on space-times admitting them, and we have studied perfect fluid solutions which admit a maximal group G_4 of isometries acting multiply transitively, together with a kinematic self-similarity.

In this case, the solutions are LRS and the maximal group G_4 acts on three-dimensional non-null orbits. Their metrics are given by the equations (1) to (3). With the exception of (1) with $k = +1$, that does not admit a simply transitive subgroup G_3 of isometries, these metrics are all special cases of the Bianchi models (see table 11.1 in [7] for subgroups G_3 on V_3 occurring in solutions with multiply-transitive groups). The possible G_3 are: for (1), $k = 0$, G_3I or G_3VII_0 ; for (1), $k = -1$, G_3III ; for (2), $k = +1$, G_3IX ; for (2), $k = 0$, G_3II ; for (2), $k = -1$, G_3VIII or G_3III ; and for (3), G_3V or G_3VII_h (see [7] chapters 11, 12 and 29 for a review of perfect fluid solutions).

We have examined the canonical line-elements of the space-times admitting a maximal group G_4 acting on three-dimensional non-null orbits, one by one, for the possible admission of the additional symmetry and then, we have restricted the source to be a perfect fluid with an arbitrary equation of state.

Many cases are empty of solutions with a maximal group G_4 , and when they exist, the proper-KSS vector usually becomes a homothetic vector and hence, the perfect fluid satisfies a linear equation of state (i.e., $p = (\gamma - 1)\mu$, where γ is a constant).

In this paper, we explicitly give *all* the homothetic solutions admitting a maximal group G_4 of isometries (see tables 11-14) along with the kinematical quantities characterizing the fluid. We realize that for the case (ii.c) the perfect fluid solutions have a non-vanishing component of the vorticity tensor. This result rejects some speculations that kinematic self-similar models were vorticity free. We notice that in this particular case, the isometry algebra acts on three-dimensional timelike orbits and therefore, the solutions are stationary.

Non-homothetic kinematic self-similar solutions are only possible for cases (i) and (iii). For case (i), the orbits associated with the kinematic self-similar and isometric algebra, respectively, must necessarily coincide. For the metric (i) with $\epsilon = -1$, the only non-homothetic solution corresponds to $k = 0$ (i.e., plane symmetry). The solution is given by equation (26). In this case, the fluid satisfies a barotropic equation of state that is not necessarily linear. For a linear equation of state, with $\gamma \neq 2$, the solution (27) was found previously by Stewart and Ellis [3], and the stiff case ($p = \mu$) corresponds to a homothetic solution. For the metric (i) with $\epsilon = +1$, any pair of functions $A(t)$ and $B(t)$ satisfying equation (29) will represent a kinematic self-similar solution.

In case (iii), the kinematic self-similar orbits are four-dimensional and the fluid flow is restricted to be non-comoving.

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